



Quintic B-spline polynomial for Solving Bagely-Torvik Fractional Differential Problems

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Abstract

In this paper, we use Quintic B-spline function to drive some error analysis of fractional derivatives and finding numerical solution for certain type fractional differential equations. It is know that a class of control points related to the modulus of continuity can be found by matrices operations, and discuss the error estimations for numerical solutions finding by using quantic B-spline to be in good agreement with the exact solutions. Results are shown graphically and illustrated the method by several examples.

Key Words: Quintic B-spline, control points, Fractional derivative, Taylor series, Convergence analysis

1. Introduction

Recently, fractional calculus has very important roles in modelling many problems in physical science [1-3]. Fractional differential equations are usually difficult to solve analytically, various limited method have been proposed for finding numerical solution to these equations [4- 7]. B-Spline polynomial function used by many anthers for numerical analysis and approximate solution of boundary value problems [8, 9]. Till now there are a few simple attempts to use B-Spline for solving fractional calculus. The approximation theory of spline polynomial is very good field of solving differential equations, when numerical algorithms aspect of behavior are considered. Among of the several class derivatives of spline functions, the polynomial spline has been accepted the more interesting because define a basic of B-spline [13-18, 23, 25, 28], which can be precisely defined.

In this paper we study quintic B-spline polynomial function for solving a for certain type fractional differential equations, which apply the quintic B-spline method for approximation solution for a Bagely-Torvik fractional differential equation

$$a(t)y'' + [b(t)D^{(\alpha)} + c(t)]y = f(t), \quad t \in [a, b], \quad \alpha \text{ is positive non-integer } \alpha \in R^+ / Z^+. \quad (1)$$

where the function $a(t), b(t), c(t)$ and $f(t)$, are sufficiently smooth real valued functions as [10, 11, 27], in particular, the $(\alpha = 1/2)$ and $(\alpha = 3/2)$ order derivatives describe the frequency- dependent damping materials of real physical systems. The paper constructed as follows, the next section give preliminaries and basic definition and theorem necessary to our work. In the third section derivations and modeled approximation polynomial by using quintic B-spline function for finding numerical solution of the equation (1), and estimate the error converges for solving quantic B-spline polynomial function. Finally in the last section numerical results are included, to show the efficiency and advantages, we apply our new method for solving numerical solution of several fractional problems.

2 Preliminaries and Basic Definitions

In this section, we define some basic definitions as follows:

Definition 2.1 [21]

The Caputo fractional derivative of order $\alpha > 0$ is defined

$${}_a D_a^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(n-a)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi, & n-1 < \alpha < n \in \mathbb{N}; \\ \frac{d^n}{dx^n} f(x), & \alpha = n \in \mathbb{N}. \end{cases}$$

where $f \in C^n[a, b]$.

Definition 2.2 [20, 22]

Suppose that $D_a^{k\alpha} f(x) \in C[a, b]$ for $k = 0, 1, \dots, n+1$ where $0 < \alpha \leq 1$, then we have the Taylor Series expansion about $x = \tau$, as follows

$$f(x) = \sum_{i=0}^n \frac{(x-\tau)^{i\alpha}}{\Gamma(i\alpha+1)} D_a^{i\alpha} f(\tau) + \frac{(D_a^{(n+1)\alpha} f)(\xi)}{\Gamma((n+1)\alpha+1)} (x-\tau)^{(n+1)\alpha} \quad \text{With } a \leq \xi \leq x, \text{ for all } x \in (a, b],$$

where $D_a^{k\alpha} = D_a^\alpha \cdot D_a^\alpha \dots D_a^\alpha$ (k times).

Definition 2.3 [23]

The modulus of continuity of a function f continuous on a segment $[a, b]$, $f \in C[a, b]$ is a function $\omega(f, t) = \omega(f, t)$ defined for $t \in [0, b-a]$ by the relation

$$\omega(f, t) = \omega(f, t; \delta) = \max_{|t-x| \leq \delta} |f(t) - f(x)|.$$

Definition 2.4 [6, 7, 8, 12]

Suppose that us consider a partition Δ_N , on the interval $[a, b]$ is divided into n subinterval using the grids $x_i = a + ih, i = 0, 1, 2, \dots, n$, where $b = a + nh$.

Given Δ_N , a piecewise polynomial function S on the interval $[a, b]$ is called a spline of degree k if $S \in C^{k-1}[a, b]$ and S is a polynomials of degree at most k on each sub-interval $[x_i, x_{i+1}]$. Let $S_k(\Delta_k)$ denoted the set of all polynomials of degree k associated with Δ_N . This set is a linear space with respect to Δ_N of dimension $N + k$. Now that we have defined spline functions, we introduced a special kind of spline function called B-spline of degree 5, [12, 13]

Quintic B-spline is defined by

$$B_{i,5}(x) = \frac{1}{120h^5} \begin{cases} (x - x_{i-3})^5, & \text{if } x \in [x_{i-3}, x_{i-2}) \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5, & \text{if } x \in [x_{i-2}, x_{i-1}) \\ (x - x_{i-3})^5 - 6(x - x_{i-2})^5 + 15(x - x_{i-1})^5, & \text{if } x \in [x_{i-1}, x_i) \\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5 + 15(x_{i+1} - x)^5, & \text{if } x \in [x_i, x_{i+1}) \\ (x_{i+3} - x)^5 - 6(x_{i+2} - x)^5, & \text{if } x \in [x_{i+1}, x_{i+2}) \\ (x_{i+3} - x)^5, & \text{if } x \in [x_{i+2}, x_{i+3}) \\ 0, & \text{other wise} \end{cases} \quad (2)$$

Form the fractional derivative of the quantic B-spline equation, can be find $B_{i,5}^{(k)}(x)$, where $k = \frac{1}{2}$, we obtain

$$B_{i,5}^{(\frac{1}{2})}(x) = \frac{1}{h^5} \begin{cases} \rho(x - x_{i-3})^{\frac{9}{2}}, & \text{if } x \in [x_{i-3}, x_{i-2}) \\ \rho(x - x_{i-3})^{\frac{9}{2}} - \sigma(x - x_{i-2})^{\frac{9}{2}}, & \text{if } x \in [x_{i-2}, x_{i-1}) \\ \rho(x - x_{i-3})^{\frac{9}{2}} - \sigma(x - x_{i-2})^{\frac{9}{2}} + \tau(x - x_{i-1})^{\frac{9}{2}}, & \text{if } x \in [x_{i-1}, x_i) \\ -\rho(x_{i+3} - x)^{\frac{9}{2}} + \sigma(x_{i+2} - x)^{\frac{9}{2}} - \tau(x_{i+1} - x)^{\frac{9}{2}}, & \text{if } x \in [x_i, x_{i+1}) \\ -\rho(x_{i+3} - x)^{\frac{9}{2}} + \sigma(x_{i+2} - x)^{\frac{9}{2}}, & \text{if } x \in [x_{i+1}, x_{i+2}) \\ -\rho(x_{i+3} - x)^{\frac{9}{2}}, & \text{if } x \in [x_{i+2}, x_{i+3}) \\ 0, & \text{other wise} \end{cases} \quad (3)$$

where $\rho = 0.0191048325$, $\sigma = 0.114628995$ and $\tau = 0.2865724875$

$B_i(x)$ ($i = -2, 0, \dots, n + 2$) are linearly independent, and they form the basis splines of $S_3(I)$, $B_i^k(x) = B_i^k(x + h)$, ($i = -2, 0, \dots, n; k = 0, 1, 2$), interpolating all points, $\sum_{i=-2}^{n+2} B_i(x) = 1$ ($x \in [a, b]$). (4)

3. Convergence analysis of the method

In this section, we proof some lemma and theorem using quintic B-spline of the fractional derivatives.

Lemma 3.1 [12] For $j = 0, 1, 2, \dots, n$, let $y_i = y_i(x)$, we have $I_j = \frac{E-I}{D} y_i$, where E, I and D , are operators.

Lemma 3.2 Let $S(x)$ be the quintic B-spline polynomial obtained by (1) and (4) for $y(x)$. for $j = 0, 1, 2, \dots, n$, we obtain the following relations:

$$S_j = \frac{6}{hD} \left(\frac{-E^{-2} - 25E^{-1} - 40I + 40E + 25E^2 + E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j, \quad (5)$$

$$S'_j = \frac{30}{h^2D} \left(\frac{E^{-2} + 9E^{-1} - 10I - 10E + 9E^2 + E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j, \quad (6)$$

$$S''_j = \frac{120}{h^3D} \left(\frac{-E^{-2} - E^{-1} - 8I - 10E + E^2 + E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j, \quad (7)$$

$$S'''_j = \frac{360}{h^4D} \left(\frac{E^{-2} - 3E^{-1} + 2I + 2E - 3E^2 + E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j, \quad (8)$$

$$S^{(4)}_j = \frac{720}{h^5 D} \left(\frac{-E^{-2} + 5E^{-1} - 10I + 10E - 5E^2 + E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j, \quad (9)$$

$$S^{(1/2)}_j = \frac{720}{h^{3/2} D} \left(\frac{0.019048325E^{-2} + 0.2986156917E^{-1} + 0.0554841863I - 0.6908122197E + 0.2986156917E^2 + 0.019048325E^{-2}E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j. \quad (10)$$

where E, I and D, are operators.

Proof: The quantic B-spline interpolation problem is start follows

$$\int_{x_j}^{x_{j+1}} S(x) dx = I_j = \int_{x_j}^{x_{j+1}} y(x) dx \quad (j = 0, 1, \dots, n - 1) \quad (11)$$

$$S(x_0) = y_0, S(x_n) = y_n \quad (12)$$

From equation (3), we obtain the desired integro quintic B-spline with a known control points as:

$$I_j = \int_{x_j}^{x_{j+1}} \sum_{i=-2}^{n+2} C_i B_i(x) dx \text{ and } I_j = \int_{x_j}^{x_{j+1}} \sum_{i=j-2}^{n+2} C_i B_i(x) dx \quad (13)$$

$$I_j = \int_{x_j}^{x_{j+1}} C_{j-1} B_{j-1}(x) dx + \int_{x_j}^{x_{j+1}} C_j B_j(x) dx + \int_{x_j}^{x_{j+1}} C_{j+1} B_{j+1}(x) dx + \int_{x_j}^{x_{j+1}} C_{j+2} B_{j+2}(x) dx$$

From the condition of differential equations, we know that $B_i(x_{i-2})$

Using equations (3) and (7), with the boundary conditions of equation (2), we have

$$\begin{aligned} & \frac{h}{720} (S_{j-2} + 57S_{j-1} + 302S_j + 302S_{j+1} + 57S_{j+2} + S_{j+3}) = \\ & \frac{h}{720} \left[\frac{1}{120} (C_{j-4} + 26C_{j-3} + 66C_{j-2} + 26C_{j-1} + C_j) + \right. \\ & \quad \frac{1}{120} \times (57) \times (C_{j-3} + 26C_{j-2} + 66C_{j-1} + 26C_j + C_{j+1}) + \\ & \quad \frac{1}{120} \times (302) \times (C_{j-2} + 26C_{j-1} + 66C_j + 26C_{j+1} + C_{j+2}) + \\ & \quad \frac{1}{120} \times (302) \times (C_{j-1} + 26C_j + 66C_{j+1} + 26C_{j+2} + C_{j+3}) + \\ & \quad \frac{1}{120} \times (57) \times (C_j + 26C_{j+1} + 66C_{j+2} + 26C_{j+3} + C_{j+4}) + \\ & \quad \left. \frac{1}{120} (C_{j+1} + 26C_{j+2} + 66C_{j+3} + 26C_{j+4} + C_{j+5}) \right] \end{aligned}$$

Then

$$\frac{h}{720} (S_{j-2} + 57S_{j-1} + 302S_j + 302S_{j+1} + 57S_{j+2} + S_{j+3}) = \frac{1}{120} (I_{j-2} + 26I_{j-1} + 66I_j + 26I_{j+1} + I_{j+2})$$

Then

$$\begin{aligned} & \frac{h}{720} (E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3) S_j = \\ & \frac{1}{120} (E^{-2} + 26E^{-1} + 66I + 26E + E^2) I_j \end{aligned}$$

Here using the lemma 2.1, we obtain

$$I_j = \sum_{i=0}^{\infty} \frac{h^{i+1}}{(i+1)!} y^{(i)}(x_j) = \frac{e^{hD} - 1}{D} y(x_j) = \frac{E - I}{D} y_j$$

$$S_j = \frac{6}{hD} \left(\frac{-E^{-2} - 25E^{-1} - 40I + 40E + 25E^2 + E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j$$

By the same way, we can find the following:

$$\frac{h}{720} (S'_{j-2} + 57S'_{j-1} + 302S'_j + 302S'_{j+1} + 57S'_{j+2} + S'_{j+3}) =$$

$$\frac{h}{720} \left[\frac{1}{24h} (-C_{j-4} - 10C_{j-3} + 10C_{j-1} + C_j) + \right.$$

$$\frac{1}{24h} \times (57) \times (-C_{j-3} - 10C_{j-2} + 10C_j + C_{j-1}) +$$

$$\frac{1}{24h} \times (302) \times (-C_{j-2} - 10C_{j-1} + 10C_{j+1} + C_{j+2}) +$$

$$\frac{1}{24h} \times (302) \times (-C_{j-1} - 10C_j + 10C_{j+2} + C_{j+3}) +$$

$$\frac{1}{24h} \times (57) \times (-C_j - 10C_{j+1} + 10C_{j+3} + C_{j+4}) +$$

$$\left. \frac{1}{24h} (-C_{j+1} - 10C_{j+2} + 10C_{j+4} + C_{j+5}) \right],$$

then we obtain

$$\frac{h}{720} (S'_{j-2} + 57S'_{j-1} + 302S'_j + 302S'_{j+1} + 57S'_{j+2} + S'_{j+3}) = \frac{1}{24h} (-I_{j-2} - 10I_{j-1} + 10I_{j+1} + I_{j+2})$$

Then

$$\frac{h}{720} (E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3) S'_j =$$

$$\frac{1}{24h} (-E^{-2} - 10E^{-1} + 10E + E^2) I_j$$

Then, here using the lemma 2.1, we obtain

$$I_j = \sum_{i=0}^{\infty} \frac{h^{i+1}}{(i+1)!} y^{(i)}(x_j) = \frac{e^{hD} - 1}{D} y(x_j) = \frac{E - I}{D} y_j$$

$$S'_j = \frac{30}{h^2 D} \left(\frac{E^{-2} + 9E^{-1} - 10I - 10E + 9E^2 + E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j$$

By a same way we find S''_j

$$S''_j = \frac{120}{h^3} \left(\frac{E^{-2} + 2E^{-1} - 6I - 2E + E^2}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j$$

Here using the lemma 2.1, we obtain

$$I_j = \sum_{i=0}^{\infty} \frac{h^{i+1}}{(i+1)!} y^{(i)}(x_j) = \frac{e^{hD} - 1}{D} y(x_j) = \frac{E - I}{D} y_j$$

$$S''_j = \frac{120}{h^3 D} \left(\frac{-E^{-2} - E^{-1} - 8I - 10E + E^2 + E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j$$

By a same way we find S'''_j

$$S'''_j = \frac{360}{h^4} \left(\frac{-E^{-2} + 2E^{-1} - 2E + E^2}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j$$

Here using the lemma 2.1, we obtain

$$I_j = \sum_{i=0}^{\infty} \frac{h^{i+1}}{(i+1)!} y^{(i)}(x_j) = \frac{e^{hD} - 1}{D} y(x_j) = \frac{E - I}{D} y_j$$

$$S'''_j = \frac{360}{h^4 D} \left(\frac{E^{-2} - 3E^{-1} + 2I + 2E - 3E^2 + E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j$$

By a same way we find S''_j

$$S^{(4)}_j = \frac{720}{h^5} \left(\frac{E^{-2} - 4E^{-1} + 6I - 4E + E^2}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j$$

Here using the lemma 2.1, we obtain

$$I_j = \sum_{i=0}^{\infty} \frac{h^{i+1}}{(i+1)!} y^{(i)}(x_j) = \frac{e^{hD} - 1}{D} y(x_j) = \frac{E - I}{D} y_j$$

$$S^{(4)}_j = \frac{720}{h^5 D} \left(\frac{-E^{-2} + 5E^{-1} - 10I + 10E - 5E^2 + E^3}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right) y_j$$

By a same way we find $S^{(\frac{1}{2})}$

$$\frac{h}{720} (S'_{j-2} + 57S'_{j-1} + 302S'_j + 302S'_{j+1} + 57S'_{j+2} + S'_{j+3}) =$$

$$\frac{1}{\sqrt{n}} (-0.019048325I_{j-2} - 0.3176640167I_{j-1} - 0.373148203I_j + 0.3176640167I_{j+1} + 0.019048325I_{j+2})$$

Then

$$\frac{h}{720} (E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3) S'_j =$$

$$\frac{1}{\sqrt{n}} (-0.019048325E^{-2} - 0.3176640167E^{-1} - 0.373148203I + 0.3176640167E + 0.019048325E^2)$$

Similarly from lemma 2.1, we obtain

$$S^{(1/2)}_j = \frac{720}{h^{3/2}} \left(\frac{-0.019048325E^{-2} - 0.3176640167E^{-1} - 0.373148203I + 0.3176640167E + 0.019048325E^2}{E^{-2} + 57E^{-1} + 302I + 302E + 57E^2 + E^3} \right)$$

Here using the lemma 2.1, we obtain

$$I_j = \sum_{i=0}^{\infty} \frac{h^{i+1}}{(i+1)!} y^{(i)}(x_j) = \frac{e^{hD} - 1}{D} y(x_j) = \frac{E - I}{D} y_j$$

Hence, equations (4)-(6) are obtained.

Theorem 3.3 Consider the function $y(x)$ of class $C^\infty [a, b]$ and quintic B-Spline polynomial $S(x)$, with fractional derivative obtained by (2) and (3) for $j = 0, 1, 2, \dots, n$, then

$$S(x) = y(x_j) - \frac{15.8333333333}{720} h^5 y^{(5)} + o(h^6) \tag{14}$$

$$S' = y'(x_j) - \frac{1.422466667}{720} h^6 y^{(6)} + o(h^7) \tag{15}$$

$$S_j^{(1/2)} = y^{(1/2)}(x_j) + \mu h^2 y^{(3/2)} + o(h^3) \tag{16}$$

$$S''_j = y''(x_j) - \frac{56.5167}{720} h^4 y^{(6)} + o(h^5) \tag{17}$$

$$S_j''' = y'''(x_j) - \frac{3}{720}h^4y^{(7)} + o(h^5) \tag{18}$$

$$S_j^{(4)} = y^{(4)}(x_j) - \frac{60}{720}h^2y^{(6)} + o(h^3) \tag{19}$$

where the constant $\mu = 0.373148203h^{-\frac{1}{2}} + 0.7115213315h^{\frac{1}{2}}y'(x_j)$

Proof: First to proof (14), the proofs for the others are similar and omitted by (4) and lemma 2.1, Suppose that $E = e^u$ and $u = hD$, and

$$e^{3u} = 1 + 3u + \frac{9u^2}{2} + \frac{27u^3}{6} + \frac{81u^4}{24} + \frac{243u^5}{120} + \frac{729u^6}{720} + \frac{2187u^7}{5040} + \frac{6561u^8}{40320} + \frac{19683u^9}{362880} + \dots$$

$$S_j = \left(\frac{720 + 360u + 300u^2 + 120u^3 + 57u^4 + 3.166666666u^5 + \dots}{720 + 360u + 300u^2 + 120u^3 + 57u^4 + 19u^5 + 6.6666u^6 + \dots} \right) y_j$$

$$S_j = \left(1 + \frac{3.167 - 19}{720}u^6 + cu^7 \right) y_j$$

where c is a certain constant and the same technique in [12], we obtain

$$S(x) = y(x_j) - \frac{15.8333333333}{720}h^5y^{(5)} + o(h^6)$$

From equations (6) , with apply lemma 2.1, we have

$$S'_j = D \left(\frac{720 + 360u + 300u^2 + 120u^3 + 57u^4 + 19u^5 + 5.2442u^6 + \dots}{720 + 360u + 300u^2 + 120u^3 + 57u^4 + 19u^5 + 6.6666u^6 + \dots} \right) y_j,$$

$$S'_j = \left(1 + \frac{5.2442 - 6.6666}{720}u^6 + cu^7 \right) Dy_j$$

$$S' = y'(x_j) - \frac{1.422466667}{720}h^6y^{(6)} + o(h^7)$$

From equation (7), with apply lemma 2.1, we have

$$S''_j = \frac{120}{h^3D} \left(\frac{-e^{-2u} - e^{-u} - 8 - 10e^u + e^{2u} + e^{3u}}{e^{-2u} + 57e^{-u} + 302 + 302e^u + 57e^{2u} + e^{3u}} \right) y_j$$

$$S''_j = \left(1 + \frac{0.4833 - 57}{720}u^4 + cu^5 \right) D^2y_j$$

From equation (8), with apply lemma 2.1, we have

$$S'''_j = \frac{360}{h^4D} \left(\frac{e^{-2u} - 3e^{-u} + 2 + 2e^u - 3e^{2u} + e^{3u}}{e^{-2u} + 57e^{-u} + 302 + 302e^u + 57e^{2u} + e^{3u}} \right) y_j \tag{20}$$

$$S'''_j = \left(1 + \frac{54 - 57}{720}u^4 + cu^5 \right) D^3y_j,$$

$$S_j''' = y'''(x_j) - \frac{3}{720}h^4y^{(7)} + o(h^5)$$

From equation (9), with apply lemma 2.1, then

$$S^{(4)}_j = \frac{720}{h^5D} \left(\frac{-e^{-2u} + 5e^{-u} - 10 + 10e^u - 5e^{2u} + e^{3u}}{e^{-2u} + 57e^{-u} + 302 + 302e^u + 57e^{2u} + e^{3u}} \right) y_j$$

$$S^{(4)}_j = \left(1 + \frac{240 - 300}{720}u^2 + cu^5 \right) D^4y_j$$

$$S_j^{(4)} = y^{(4)}(x_j) - \frac{60}{720}h^2y^{(6)} + o(h^3)$$

From equation (10), with apply lemma 2.1, then

$$S^{(1/2)}_j = \frac{720}{h^{3/2}D} \left(\frac{0.019048325e^{-2u} + 0.2986156917e^{-u} + 0.0554841863 - 0.6908122197e^u + 0.2986156917e^{2u} + e^{3u}}{e^{-2u} + 57e^{-u} + 302 + 302e^u + 57e^{2u} + e^{3u}} \right) y_j$$

$$S^{(1/2)}_j = S^{(1/2)}_j = \frac{1}{h^{1/2}} (-.373148203 + 0.7115213315u + cu^2) y_j$$

$$S^{(1/2)}_j = \left(-.373148203h^{-\frac{1}{2}} + 0.7115213315h^{\frac{1}{2}}y'(x_j) + ch^{\frac{3}{2}}y''(x_j) \right)$$

$$S_j^{(1/2)} = y^{(1/2)}(x_j) + \mu h^2y^{(3/2)} + o(h^3) \tag{21}$$

where the constant $\mu = -.373148203h^{-\frac{1}{2}} + 0.7115213315h^{\frac{1}{2}}y'(x_j)$

Theorem 3.4 Consider the function $y(x)$ of class $C^\infty[a, b]$ and $S(x)$ be the Quintic B-spline polynomial have

$$\int_{x_j}^{x_{j+1}} s(x)dx = I_j = \int_{x_j}^{x_{j+1}} y(x)dx, j = 0, 1, 2, n - 1 \text{ and}$$

$$S(x_0) = y_0, S(x_1) = y_1, \dots, S(x_{n-1}) = y_{n-1}, S(x_n) = y_n$$

We have

$$\|y^{(\alpha k)}(x) - s^{(\alpha k)}(x)\|_\infty = o(h^{3-\alpha k}), k = 0, 1, 2, 3.$$

where $\|f(x)\|_\infty = \max_{a \leq x \leq b} |f(x)|, 0 < \alpha \leq 1.$

Proof: First try to show $\|y^{(4)}(x) - S^{(4)}(x)\|_\infty = o(h^2).$ since $S(x)$ is a quintic B-Spline hence $S''(x)$ is a piecewise continuous function on $[a, b],$ for $j = 1, 2, \dots, n.$

As [12], suppose that $S^{(4)}(x)$ denoted the restriction of $S^{(4)}(x)$ over $[x_{j-1}, x_j],$ then

$$S^{(4)}(x) = S^{(4)}(x_{j-1}) \frac{x_j - x}{h} + S^{(4)}(x_j) \frac{x - x_{j-1}}{h}$$

Now define another linear function $g(x)$ on $[x_{j-1}, x_j]$ as follows

$$g(x) = y''(x_{j-1}) \frac{x_j - x}{h} + y''(x_j) \frac{x - x_{j-1}}{h}$$

Clearly $g(x)$ is a linear interpolation of $y^{(4)}$

$$\begin{aligned} \|S^{(4)}(x) - g(x)\|_\infty &= \max_{x_{j-1} \leq x \leq x_j} \left| \left(S^{(4)}(x_{j-1}) - y^{(4)}(x_{j-1}) \right) \frac{x_j - x}{h} + \left(S^{(4)}(x_j) - y^{(4)}(x_j) \right) \frac{x - x_{j-1}}{h} \right| \\ &= o(h^2) \end{aligned}$$

$$\begin{aligned} \|S^{(4)}(x) - y^{(4)}(x)\|_\infty &= \|S^{(4)}(x) - g(x) + g(x) - y^{(4)}(x)\|_\infty \\ &\leq \|S^{(4)}(x) - g(x)\|_\infty + \|g(x) - y^{(4)}(x)\|_\infty \\ &\leq o(h^2) + o(h^2) = o(h^2). \end{aligned}$$

and using theorem 4.2 in [12],

$$\begin{aligned} |S'''(x) - y'''(x)| &= \int_{x_{j-1}}^x S^{(4)}(t)dt + S'''(x_{j-1}) - \int_{x_{j-1}}^x y^{(4)}(t)dt + y'''(x_{j-1}) \\ &= \int_{x_{j-1}}^x [S^{(4)}(t) - y^{(4)}(t)]dt + [S'''(x_{j-1}) - y'''(x_{j-1})] \\ &= o(h^3) + o(h^4) \\ &= o(h^3) \end{aligned}$$

$$\begin{aligned}
 |S''(x) - y''(x)| &= \int_{x_{j-1}}^x S'''(t)dt + S''(x_{j-1}) - \int_{x_{j-1}}^x y'''(t)dt + y''(x_{j-1}) \\
 &= \int_{x_{j-1}}^x [S'''(t) - y'''(t)]dt + [S''(x_{j-1}) - y''(x_{j-1})] \\
 &= o(h^4) + o(h^4) \\
 &= o(h^4)
 \end{aligned}$$

$$\begin{aligned}
 |S'(x) - y'(x)| &= \int_{x_{j-1}}^x S''(t)dt + s'(x_{j-1}) - \int_{x_{j-1}}^x y''(t)dt + y'(x_{j-1}) \\
 &= \int_{x_{j-1}}^x [S''(t) - y''(t)]dt + [s'(x_{j-1}) - y'(x_{j-1})] \\
 &= o(h^5) + o(h^6) \\
 &= o(h^5)
 \end{aligned}$$

$$\begin{aligned}
 |S(x) - y(x)| &= \int_{x_{j-1}}^x S'(t)dt + s(x_{j-1}) - \int_{x_{j-1}}^x y'(t)dt + y(x_{j-1}) \\
 &= \int_{x_{j-1}}^x [S'(t) - y'(t)]dt + [s(x_{j-1}) - y(x_{j-1})] \\
 &= o(h^6) + o(h^6) \\
 &= o(h^6)
 \end{aligned}$$

and

$$\begin{aligned}
 \|y^{(\alpha)} - S^{(\alpha)}\| &\leq \frac{h^2}{4} h^\alpha \sqrt{\pi} W \alpha \\
 &\leq \frac{h^2}{4} \|D^2 D^\alpha f\|
 \end{aligned}$$

$\|y^{(\alpha)} - S^{(\alpha)}\| \leq \frac{h^2}{4} \|D^{(5/2)} f\|$, for $\alpha = \frac{1}{2}$, obtain the following

$$\left| g^{(r)}(x) - p_{2m-1}^{(r)}(\alpha) \right| \leq \frac{h^r [(x - x_i)(x_{i+1} - x)]^{m-r} G}{r! (2m - 2r)!}, \quad G = \max |g^{(2m)}(x)|$$

$m = 1, g = f^{\frac{1}{2}}, p_1 = S^{\frac{1}{2}}$, and using [24, 26, 28], we have

$$\begin{aligned}
 \left| S^{(r+\frac{1}{2})}(x) - y^{(r+\frac{1}{2})}(x) \right| &\leq \frac{h^r [(x - x_i)(x_{i+1} - x)]^{1-r}}{r! (2 - 2r)!} \|y^{(3/2)}\|_\infty \\
 r = 0 \rightarrow \|S^{(1/2)}(x) - y^{(1/2)}(x)\| &\leq \frac{(x - x_i)(x_{i+1} - x)}{2!} \|y^{(3/2)}\|_\infty \\
 &\leq \frac{h^2}{2!} \|y^{(3/2)}\|_\infty \\
 \|S^{(1/2)}(x) - y^{(1/2)}(x)\| &\leq \frac{h^2}{2} \|y^{(3/2)}\|_\infty
 \end{aligned}$$

If $r = 1, 2, 3, 4$ respectively, using definition 2.4 and theorem 2.3, then we have

$$\begin{aligned}
 |S_j - y(x_j)| &< \frac{15.8333333333}{720} h^5 \omega(y^{(4)}, x) \\
 |S_j'' - y''(x_j)| &< \frac{56.5167}{720} h^4 \omega(y^{(4)}, x) \\
 |S_j''' - y'''(x_j)| &< \frac{3}{720} h^4 \omega(y^{(4)}, x)
 \end{aligned}$$

$$|S_j^{(4)} - y^{(4)}(x_j)| < \frac{60}{720} h^2 \omega(y^{(4)}, x)$$

where $\omega(y^{(4)}, x)$ is the modulus of continuity.

4 Numerical Approximations and Discussion

To verify the applicability of cubic B-Spline to solve fractional Bagley-Torvik differential problem in [9, 10, 11, 27], and to show the our approximate solution approach the nature solution,

$$a(x)y'' + b(x)D^{(1/2)}y + c(x)y = f(x) \tag{22}$$

With boundary condition

$$y(a) = \alpha, \quad y(b) = \beta \tag{23}$$

$$y'(a) = \alpha', \quad y'(b) = \beta'$$

where $a(x), b(x), c(x)$ and $f(x)$ are continuous real-valued function on the interval $[a, b]$, we let $y(x)$ be a quintic spline with knots Δ . Then $y(x)$ can be written as a linear combination of $B_i(x)$,

$$y(x) = \sum_{i=-2}^{N+2} C_i B_i(x) \tag{24}$$

Where the constant C_i are determined and $B_i(x)$ are defined in equation (3).

It is necessary that (14) satisfy (12 and 13) at $x = x_i$ where x_i is an interior point that is

$$a(x_i)y''(x_i) + b(x_i)D^{(1/2)}y''(x_i) + c(x_i)y(x_i) = f(x_i)$$

and the boundary conditions are

$$y(a) = \alpha \text{ for } x_0 = a, \quad y(b) = \beta \text{ for } x_N = b \tag{25}$$

From (24) we obtained

$$\begin{aligned} y(x_i) &= c_{i-2}B_{i-2}(x_i) + c_{i-1}B_{i-1}(x_i) + C_i B_i(x_i) + c_{i+1}B_{i+1}(x_i) + C_{i+2}B_{i+2}(x_i) + C_{i+3}B_{i+3}(x_i) \\ y^{\frac{1}{2}}(x_i) &= C_{i-2}B_{i-2}^{\frac{1}{2}}(x_i) + C_{i-1}B_{i-1}^{\frac{1}{2}}(x_i) + C_i B_i^{\frac{1}{2}}(x_i) + C_{i+1}B_{i+1}^{\frac{1}{2}}(x_i) + C_{i+2}B_{i+2}^{\frac{1}{2}}(x_i) + C_{i+3}B_{i+3}^{\frac{1}{2}}(x_i) \end{aligned} \tag{26}$$

$$y''(x_i) = C_{i-2}B''_{i-2}(x_i) + C_{i-1}B''_{i-1}(x_i) + C_i B''_i(x_i) + C_{i+1}B''_{i+1}(x_i) + C_{i+2}B''_{i+2}(x_i) + C_{i+3}B''_{i+3}(x_i)$$

There yield

$$\begin{aligned} a(x_i)[C_{i-2}B''_{i-2}(x_i) + C_{i-1}B''_{i-1}(x_i) + C_i B''_i(x_i) + C_{i+1}B''_{i+1}(x_i) + C_{i+2}B''_{i+2}(x_i) + C_{i+3}B''_{i+3}(x_i)] + \\ b(x_i)[C_{i-2}B_{i-2}^{\frac{1}{2}}(x_i) + C_{i-1}B_{i-1}^{\frac{1}{2}}(x_i) + C_i B_i^{\frac{1}{2}}(x_i) + C_{i+1}B_{i+1}^{\frac{1}{2}}(x_i) + C_{i+2}B_{i+2}^{\frac{1}{2}}(x_i) + C_{i+3}B_{i+3}^{\frac{1}{2}}(x_i)] + \\ c(x_i)[c_{i-2}B_{i-2}(x_i) + c_{i-1}B_{i-1}(x_i) + C_i B_i(x_i) + c_{i+1}B_{i+1}(x_i) + C_{i+2}B_{i+2}(x_i) + C_{i+3}B_{i+3}(x_i)] = f(x_i) \end{aligned} \tag{27}$$

By the properties of Qunitic B-spline function, we have

$$\begin{aligned} B_{i-2}(x_i) &= \frac{1}{120} & B_{i-2}^{\frac{1}{2}}(x_i) &= \frac{-0.019048325}{\sqrt{h}} & B''_{i-2}(x_i) &= \frac{1}{6h^2} \\ B_{i-1}(x_i) &= \frac{26}{120} & B_{i-1}^{\frac{1}{2}}(x_i) &= \frac{-0.3176640167}{\sqrt{h}} & B''_{i-1}(x_i) &= \frac{1}{3h^2} \\ B_i(x_i) &= \frac{66}{120} & B_i^{\frac{1}{2}}(x_i) &= \frac{-0.3731482029}{\sqrt{h}} & B''_i(x_i) &= \frac{-1}{h^2} \end{aligned} \tag{28}$$

$$\begin{aligned}
 B_{i+1}(x_i) &= \frac{26}{120} & B_{i+1}^{\frac{1}{2}}(x_i) &= \frac{0.3176640167}{\sqrt{h}} & B''_{i+1}(x_i) &= \frac{1}{3h^2} \\
 B_{i+2}(x_i) &= \frac{1}{120} & B_{i+2}^{\frac{1}{2}}(x_i) &= \frac{0.019048325}{\sqrt{h}} & B''_{i+2}(x_i) &= \frac{1}{6h^2} \\
 B_{i+3}(x_i) &= 0 & B_{i+3}^{\frac{1}{2}}(x_i) &= 0 & B''_{i+3}(x_i) &= 0
 \end{aligned}$$

Form equations (25) and (18) we obtain

$$\begin{aligned}
 a(x_i)[\frac{1}{6h^2}C_{i-2} + \frac{1}{3h^2}C_{i-1} + (\frac{-1}{h^2})C_i + \frac{1}{3h^2}C_{i+1} + \frac{1}{6h^2}C_{i+2} + (0)C_{i+3}] + b(x_i)[(\frac{-0.019048325}{\sqrt{h}})C_{i-2} + (\frac{-0.3176640167}{\sqrt{h}})C_{i-1} + \\
 (\frac{-0.3731482029}{\sqrt{h}})C_i + \frac{0.3176640167}{\sqrt{h}}C_{i+1} + \frac{0.019048325}{\sqrt{h}}C_{i+2} + (0)C_{i+3}] + c(x_i)[\frac{1}{120}C_{i-2} + \frac{26}{120}C_{i-1} + \frac{66}{120}C_i + \frac{26}{120}C_{i+1} + \\
 \frac{1}{120}C_{i+2} + (0)C_{i+3}] = f(x_i) \tag{29}
 \end{aligned}$$

$$\begin{aligned}
 C_{i-2} \left(\frac{a(x_i)}{6h^2} - \frac{0.019048325b(x_i)}{\sqrt{h}} + \frac{c(x_i)}{120} \right) + C_{i-1} \left(\frac{a(x_i)}{3h^2} - \frac{0.3176640167b(x_i)}{\sqrt{h}} + \frac{13c(x_i)}{60} \right) + \\
 C_i \left(\frac{-a(x_i)}{h^2} - \frac{0.3731482029b(x_i)}{\sqrt{h}} + \frac{11c(x_i)}{20} \right) + C_{i+1} \left(\frac{a(x_i)}{3h^2} + \frac{0.3176640167b(x_i)}{\sqrt{h}} + \frac{13c(x_i)}{60} \right) + \\
 C_{i+2} \left(\frac{a(x_i)}{6h^2} + \frac{0.019048325b(x_i)}{\sqrt{h}} + \frac{c(x_i)}{120} \right) = f(x_i) \tag{30}
 \end{aligned}$$

Let that

$$\begin{aligned}
 p_i &= \frac{a(x_i)}{6h^2} - \frac{0.019048325b(x_i)}{\sqrt{h}} + \frac{c(x_i)}{120}, \quad q_i = \frac{a(x_i)}{3h^2} - \frac{0.3176640167b(x_i)}{\sqrt{h}} + \frac{13c(x_i)}{60}, \\
 r_i &= \frac{-a(x_i)}{h^2} - \frac{0.3731482029b(x_i)}{\sqrt{h}} + \frac{11c(x_i)}{20}, \quad s_i = \frac{a(x_i)}{3h^2} + \frac{0.3176640167b(x_i)}{\sqrt{h}} + \frac{13c(x_i)}{60}, \\
 t_i &= \frac{a(x_i)}{6h^2} + \frac{0.019048325b(x_i)}{\sqrt{h}} + \frac{c(x_i)}{120} \tag{31}
 \end{aligned}$$

Then

$$p_i C_{i-2} + q_i C_{i-1} + r_i C_i + s_i C_{i+1} + t_i C_{i+2} = f(x_i) \tag{32}$$

And the boundary conditions are given below

$$\begin{aligned}
 B_{-2}(x_0) = B_{N-2}(x_n) = \frac{1}{120}, \quad B_{-1}(x_0) = B_{N-1}(x_n) = \frac{26}{120}, \\
 B_0(x_0) = B_N(x_n) = \frac{66}{120}, \quad B_1(x_0) = B_{N+1}(x_n) = \frac{26}{120}, \quad B_2(x_0) = \\
 B_{N+2}(x_n) = \frac{1}{120}, \quad B_3(x_0) = B_{N+3}(x_n) = 0 \tag{33}
 \end{aligned}$$

Also form equation (27) and equation (33), we obtain

$$\begin{aligned}
 B_{-2}(x_0)C_{-2} + B_{-1}(x_0)C_{-1} + B_0(x_0)C_0 + B_1(x_0)C_1 + B_2(x_0)C_2 + B_3(x_0)C_3 = \alpha \\
 B_{N-2}(x_n)C_{N-2} + B_{N-1}(x_n)C_{N-1} + B_N(x_n)C_N + B_{N+1}(x_n)C_{N+1} + B_{N+2}(x_n)C_{N+2} + B_{N+3}(x_n)C_{N+3} = \beta \\
 C_{-2} + 26C_{-1} + 66C_0 + 26C_1 + C_2 = 120\alpha
 \end{aligned}$$

Take the first derivative, we obtain

$$\begin{aligned}
 B'_{-2}(x_0) = B'_{N-2}(x_n) = \frac{1}{24h}, \quad B'_{-1}(x_0) = B'_{N-1}(x_n) = \\
 \frac{10}{24h}, \quad B'_0(x_0) = B'_N(x_n) = 0, \quad B'_1(x_0) = B'_{N+1}(x_n) = \\
 \frac{-10}{24h}, \quad B'_2(x_0) = B'_{N+2}(x_n) = \frac{1}{24h}, \quad B'_3(x_0) = \\
 B'_{N+3}(x_n) = 0 \tag{34}
 \end{aligned}$$

$$C_{N-2} + 26C_{N-1} + 66C_N + 26C_{N+1} + C_{N+2} = 120\beta \tag{35}$$

Also form equation (32), we obtain

$$B'_{-2}(x_0)C_{-2} + B'_{-1}(x_0)C_{-1} + B'_0(x_0)C_0 + B'_1(x_0)C_1 + B'_2(x_0)C_2 + B'_3(x_0)C_3 = \alpha' \tag{36}$$

$$B'_{N-2}(x_n)C_{N-2} + B'_{N-1}(x_n)C_{N-1} + B'_N(x_n)C_N + B'_{N+1}(x_n)C_{N+1} + B'_{N+2}(x_n)C_{N+2} + B'_{N+3}(x_n)C_{N+3} = \beta' \tag{37}$$

From equation (37), we have

$$C_{-2} + 10C_{-1} + 0C_0 - 10C_1 + C_2 = 24h'\alpha' \tag{38}$$

$$C_{N-2} + 10C_{N-1} + 0C_N - 10C_{N+1} + C_{N+2} = 24h\beta \tag{39}$$

We can write a system of $N + 2$ linear equations in $N + 2$ unknown

$$\begin{pmatrix} 1 & 26 & 66 & 26 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -10 & 0 & 10 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ p_1 & q_1 & r_1 & s_1 & t_1 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & p_2 & q_2 & r_2 & s_2 & t_2 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & p_{n-1} & q_{n-1} & r_{n-1} & s_{n-1} & t_{n-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & p_n & q_n & r_n & s_n & t_n \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 26 & 66 & 26 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & -10 & 0 & 10 & 1 \end{pmatrix} \begin{pmatrix} c_{-2} \\ c_{-1} \\ c_0 \\ c_1 \\ \vdots \\ c_{N-1} \\ c_N \\ c_{N+1} \\ c_{N+2} \end{pmatrix} = \begin{pmatrix} \alpha \\ \alpha' \\ f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \\ f(x_N) \\ \beta \\ \beta' \end{pmatrix} \tag{40}$$

The piecewise quintic B-Spline function from equations (12) and (13) are obtained using (14), where the constant coefficient C_i satisfy the system defined in (24). We will consider some numerical examples demonstrating the solution using quintic B-Spline methods are illustrated; all calculations are implemented with maple programming.

Example 3.1 [10, 11] Consider the fraction boundary value problem:

$$y''(x) + 0.5D^{(1/2)} y(x) + y(x) = 3 + x^2 \left(\frac{x^{-1/2}}{\Gamma(5/2)} + 1 \right)$$

$$y(0) = 1, \quad y(0.5) = 1.25$$

The exact solution of equation (50) is $y(x) = x^2 + 1$

Using equations (26), (27) and (40) with maple programming, we obtain the following

$$S(x) = \begin{cases} 1.000000 - 1.0667 \times 10^{-8}x + 1.41978x^2 - 181.28881x^3 + 3144.4986x^4 - 42579.3049x^5, & 0 \leq x \text{ and } x < 0.05 \\ 1.0021181 - 0.211810x + 9.892216x^2 - 350.9374x^3 + 4838.984931x^4 - 22727.3699x^5, & 0.05 \leq x \text{ and } x < 0.1 \\ 1.3235256 - 15.860251x + 305.98367x^2 - 2974.3046x^3 + 14580.3481x^4 - 28708.204x^5, & 0.1 \leq x \text{ and } x < 0.15 \\ 3.087315 - 70.0599x + 963.77955x^2 - 6260.9467x^3 + 20942.502x^4 - 28004.438x^5, & 0.15 \leq x \text{ and } x < 0.2 \\ 10.4358 - 235.8524x + 2355.768x^2 - 11762.9772x^3 + 29387.872x^4 - 29370.199x^5, & 0.2 \leq x \text{ and } x < 0.25 \\ 30.595935 - 592.23401x + 4738.6145x^2 - 18953.3332x^3 + 37916.008x^4 - 30340.133x^5, & 0.25 \leq x \text{ and } x < 0.3 \\ 78.98233 - 1299.5357x + 8660.85301x^2 - 28855.035x^3 + 48073.995x^4 - 32036.2428x^5, & 0.3 \leq x \text{ and } x < 0.35 \\ 170.593 - 2424.9564x + 13869.80x^2 - 39664.3162x^3 + 56726.7797x^4 - 32454.66178x^5, & 0.35 \leq x \text{ and } x < 0.4 \\ 401.1344 - 4990.1102x + 24886.347x^2 - 62036.4794x^3 + 77307.405x^4 - 38525.2154x^5, & 0.4 \leq x \text{ and } x < 0.45 \\ -396.35021 + 4049.4951x - 17682.997x^2 + 38545.366x^3 - 41929.32x^4 + 18208.4523x^5, & 0.45 \leq x \text{ and } x < 0.5 \end{cases}$$

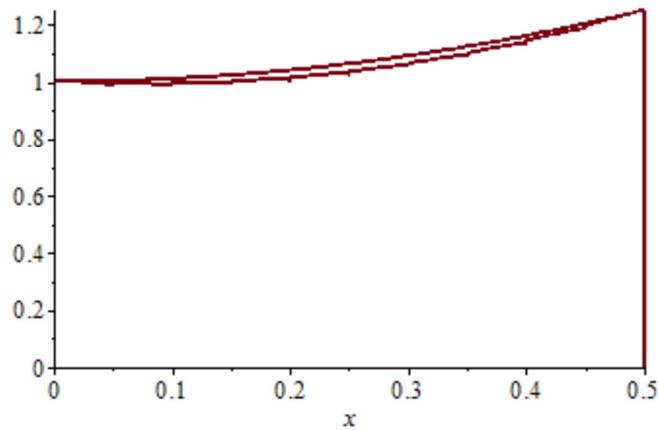


Figure 1: Graph of approximate solution $S(x)$, and $y(x)$ exact solution of the example 3.1

Table 1: The error estimations with fractional derivatives for the example 3.1

x	$e = S - y $	$e^{(1/2)} = S^{(1/2)} - y^{(1/2)} $	$e' = S' - y' $
0.00	0.000000000	0.000000000	0.000000001
0.02	0.000917129	0.000398942280	0.134434658
0.18	0.025082117	0.001196826841	0.186303400
0.24	0.029997340	0.001381976598	0.390823000
0.38	0.020493500	0.0017380949098	0.021630000
0.42	0.014551830	0.001828183198	0.143410000
0.50	0.000053800	0.001994711402	0.000210000

Example 3.2 [27] Consider the fractional differential equation

$$y''(x) + \frac{1}{2}D^{(1/2)}y(x) + y(x) = 4x^2(5x - 3) + \frac{1}{2}x^{\frac{7}{2}}\left(\frac{120x}{\Gamma(\frac{11}{2})} - \frac{24}{\Gamma(\frac{9}{2})}\right) + x^5 - x^4$$

$$y(0) = 0$$

$$y(0.5) = -0.03125$$

The exact solution of equation is $y(x) = x^4(x - 1)$

Using equations (16), (17) and (40) with maple programming, we obtain the following

$$S(x) = \begin{cases} 2.77 \times 10^{-14} + 4.2 \times 10^{-13}x + 0.00001463x^2 - 0.91075383x^3 + 14.78094x^4 - 66.701099x^5, & 0 \leq x \text{ and } x < 0.05 \\ -0.000025624 + 0.00256249x - 0.102285x^2 + 1.13924169x^3 - 5.7190x^4 + 11.947666x^5, & 0.05 \leq x \text{ and } x < 0.1 \\ -0.00186218 + 0.0938542x - 1.9068795x^2 + 18.7543639x^3 - 89.506397x^4 + 162.369572x^5, & 0.1 \leq x \text{ and } x < 0.15 \\ 0.000628687 - 0.002081x - 0.369613x^2 + 5.924646x^3 - 33.83427885x^4 + 62.3272956x^5, & 0.15 \leq x \text{ and } x < 0.2 \\ -0.112573668 + 2.7334506x - 26.46456x^2 + 127.996917x^3 - 311.00674x^4 + 302.15547x^5, & 0.2 \leq x \text{ and } x < 0.25 \\ -0.37493322 + 7.569274x - 61.037486x^2 + 245.72028x^3 - 495.03264x^4 + 397.9553538x^5, & 0.25 \leq x \text{ and } x < 0.3 \\ -4.4340679 + 23.905061x - 159.4110707x^2 + 531.50557x^3 - 887.08814x^4 + 591.92303x^5, & 0.3 \leq x \text{ and } x < 0.35 \\ -4.4134059 + 63.017188x - 359.909969x^2 + 1027.69636x^3 - 1468.16009x^4 + 838.7828x^5, & 0.35 \leq x \text{ and } x < 0.4 \\ -10.507393 + 131.51984x - 658.582243x^2 + 1649.116949x^3 - 2065.9928x^4 + 1035.4460x^5, & 0.4 \leq x \text{ and } x < 0.45 \\ -5.6283155 + 61.542708x - 268.7469x^2 + 585.751611x^3 - 638.14508x^4 + 277.6816749x^5, & 0.45 \leq x \text{ and } x < 0.5 \end{cases}$$

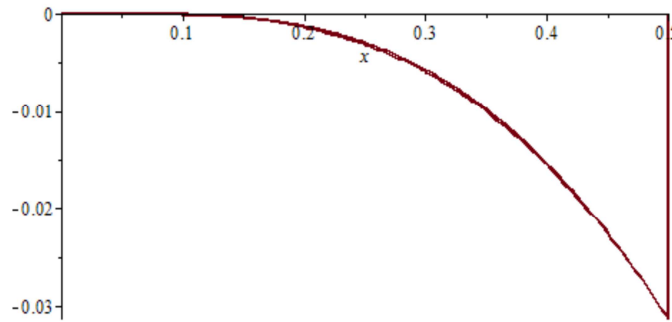


Figure 2: Graph of approximate solution $S(x)$, and $y(x)$ exact solution of the example3.2

Table 2: The maximum error with fractional derivatives for the example 3.2

x	$e = S - y $	$= S^{(1/2)} - y^{(1/2)} $	$e' = S' - y' $
0.00	0.000000000	0.000000000	0.000000000
0.02	0.000004971	$4.96056 \cdot 10^{-07}$	0.000641489
0.18	0.00004875	0.000092178919	0.003329739
0.24	0.000204049	0.000167391323	0.002003860
0.38	0.000216427	0.0003673296424	0.004194400
0.42	0.000184833	0.0004127891404	0.002775400
0.50	0.000000243	0.0004559340340	0.000000700

Example 3.3 Consider the fraction boundary value problem:

$$\frac{1}{6}y''(x) + \frac{24}{105}D^{(1/2)}y(x) + y(x) = x(x^2 + 1) + \frac{4}{\sqrt{\pi}}x^2\sqrt{x}$$

$$y(0) = 0, y(0.5) = 0.125, \quad \text{the exact solution of equation is } y(x) = x^3$$

Using equations (16), (17) and (40) with maple programing, we obtain the following

$$S(x) = \begin{cases} 1.574 \times 10^{-13} - 8.27 \times 10^{-12}x + 0.0001115x^2 - 1.078442x^3 + 35.9706633x^4 - 247.7123x^5, & 0 \leq x \text{ and } x < 0.05 \\ 0.00001683 + 0.0016183x - 0.06462349x^2 + 0.216260824x^3 + 23.023x^4 - 130.74255x^5, & 0.05 \leq x \text{ and } x < 0.10 \\ 0.0042959 - 0.2111754x + 4.07881898x^2 - 38.9695268x^3 + 196.4662x^4 - 387.682193x^5, & 0.10 \leq x \text{ and } x < 0.15 \\ 0.0501281 - 1.67649838x + 22.368023x^2 - 148.41318x^3 + 498.856536x^4 - 666.025597x^5, & 0.15 \leq x \text{ and } x < 0.20 \\ 0.348946249 - 8.7229302x + 87.178767x^2 - 1025.437x^3 + 1089.1296x^4 - 1088.785409x^5, & 0.20 \leq x \text{ and } x < 0.25 \\ 1.603219 - 32.0700769x + 256.5728x^2 - 1025.43725x^3 + 2053.16263x^4 - 1642.723107x^5, & 0.25 \leq x \text{ and } x < 0.30 \\ 5.807288 - 96.7711333x + 644.97909x^2 - 3951.168198x^3 + 3581.2745x^4 - 2386.68593x^5, & 0.30 \leq x \text{ and } x < 0.35 \\ 16.916759 - 241.790366x + 1382.410424x^2 - 3951.168198x^3 + 5649.74x^4 - 3230.6975x^5, & 0.35 \leq x \text{ and } x < 0.40 \\ 47.58969 - 594.163468x + 2966.91288x^2 - 7405.672808x^3 + 9243.93x^4 - 4614.122822x^5, & 0.40 \leq x \text{ and } x < 0.45 \\ 22.440386 + 245.3847x - 1071.593349x^2 + 2336.72813x^3 - 2540.91176x^4 - 1103.5550x^5, & 0.45 \leq x \text{ and } x < 0.50 \end{cases}$$

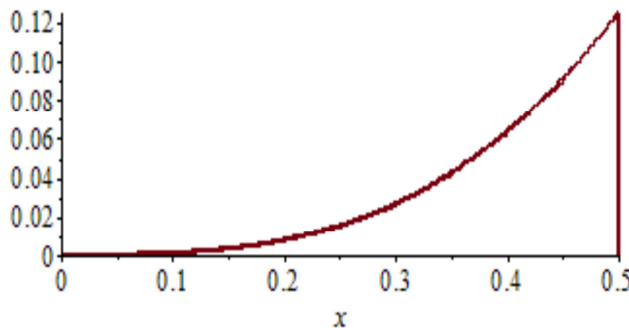


Figure 3: Graph of approximate solution $S(x)$, and $y(x)$ exact solution of the Example 3.3.

Table 3: The maximum error with fractional derivatives for the example 3.3

x	$e = S - y $	$e^{(1/2)} = S^{(1/2)} - y^{(1/2)} $	$e' = S' - y' $
0.00	0.000000000	0.0000000	0.000000000
0.02	0.000011620	0.00001595769120	0.001536772
0.18	0.000465622	0.0004308576628	0.005481300
0.24	0.000721263	0.0006633487672	0.016416750
0.38	0.000819120	0.001321601304	0.011949000
0.42	0.000647450	0.001535673886	0.001883000
0.50	0.000005000	0.001994711402	0.000027000

Conclusions

We have applied quintic B-spline polynomial to obtain the numerically solution of some fractional differential equations, which are tested their control points performance on the growth Bagley-Torvik differential equation. This method gives good results and comparable with the exact solution, a theoretical proof of the convergence of the quintic B-spline method for the fractional derivatives from class arguments. Some numerical examples were included to observed by the tables which are illustrated the error estimation of the method. Finally, the efficient could arise from the control points of the quintic B-spline that is necessary for numerical results.

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